Statistical Modeling of Complex Networks Within the ERGM family

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- call $Y \equiv [Y_{ij}]_{n \times n}$ a *sociomatrix*;
  call the graphical representation of Y a *sociogram*
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  – a $N = n(n-1)$ array of binary random variables
  – Y represents a random network with nodes the actors and edges the relationship
- The basic problem of stochastic modeling is to specify a distribution for Y i.e., $P(Y = y)$

A Framework for Network Modeling

Let $\mathcal{Y}$ be the sample space of Y e.g. $\{0,1\}^N$
Any model-class for the multivariate distribution of Y can be *parameterized* in the form:

$$P(Y = y) = \frac{\exp\{\eta^T g(y)\}}{\kappa(\eta, \mathcal{Y})} \qquad y \in \mathcal{Y}$$

Besag (1974), Bahadur (1961), Frank and Strauss (1986)

- $\eta \in \Lambda \subset R^q$ q-vector of parameters
- $g(y)$ q-vector of *network statistics*.
- For a "saturated" model-class $q = 2^{|\mathcal{Y}|} - 1$
- $\kappa(\eta)$ distribution normalizing constant

$$\kappa(\eta, \mathcal{Y}) = \sum_{y \in \mathcal{Y}} \exp\{\eta^T g(y)\}$$

Simple model-classes for social networks

Homogeneous Bernoulli graph (Renyi-Erdos model)

- Y_{ij} are independent and equally likely
  with log-odds $\eta = \text{logit}[P(Y_{ij} = 1)]$

$$P(Y = y) = \frac{e^{\eta \sum_{i,j} y_{ij}}}{\kappa(\eta)} \qquad y \in \mathcal{Y}$$

  where $q = 1$, $g(y) = \sum_{i,j} y_{ij}$, $\kappa(\eta) = [1 + \exp(\eta)]^N$
- homogeneity means it is unlikely to be proposed as a model for real phenomena

Dyad-independence models with attributes

- Y_{ij} are independent but depend on covariates dyadic $x_{k,ij}$

$$P(Y = y) = \frac{\exp\{\eta_1 g_1(y) + \eta_2 g_2(y) \ldots + \eta_K g_K(y)\}}{\kappa(\eta)} \qquad y \in \mathcal{Y}$$

$$g_k(y) = \sum_{i,j} x_{1,ij} y_{ij}, \quad k = 1, \ldots, K \qquad q = K$$

$$\text{logit}[P(Y_{ij} = 1)] = \eta_1 x_{1,ij}(y) + \eta_2 x_{2,ij} \ldots + \eta_K x_{K,ij}$$

$$\kappa(\eta) = \prod_{i,j} [1 + \exp(\sum_k \eta_k x_{k,ij})]$$

Bernoulli graphs

- Y_{ij} are independent but have arbitrary distributions

$$P(Y = y) = \frac{\exp\left\{\sum_{i,j} \eta_{ij} y_{ij}\right\}}{\kappa(\eta)} \qquad y \in \mathcal{Y}$$

$$t_{i,j}(y) = y_{ij}, \quad i,j = 1,\ldots,n \qquad q = N$$

$\eta_{ij} = \text{logit}[P(Y_{ij} = 1)]$

$$\kappa(\eta) = \prod_{i,j} [1 + \exp(\eta_{ij})]$$

- Y_{ij} can depend on dyadic covariates x_{ij}

$$\eta_{ij} = X_{ij}\beta$$

Some history of exponential family models for social networks

Holland and Leinhardt (1981) proposed a general dyad independence model

– Also an homogeneous version they refer to as the "p^1" model

$$P_\eta(Y = y) = \frac{\exp\{\rho \sum_{i<j} y_{ij}y_{ji} + \phi y_{++} + \sum_i \alpha_i y_{i+} + \sum_j \beta_j y_{+j}\}}{\kappa(\rho, \alpha, \beta, \phi)}$$

where $\eta = (\rho, \alpha, \beta, \phi)$.

– ϕ controls the expected number of edges
– ρ represent the expected tendency toward *reciprocation*
– α_i *productivity* of node i; β_j *attractiveness* of node j

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- Much related work and generalizations
  – Wasserman (1980), Fienberg, Meyer, and Wasserman (1985)

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$\Rightarrow$ analogous to nearest neighbor ideas in spatial statistics

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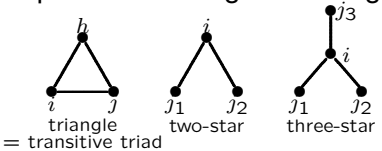
- Degree distribution: $\mathrm{d}_k(\mathbf{y}) =$ proportion of nodes of degree k in $\mathbf{y}$.
- k-star distribution: $\mathrm{s}_k(\mathbf{y}) =$ proportion of k-stars in the graph $\mathbf{y}$.
- triangles: $\mathrm{t}_1(\mathbf{y}) =$ proportion of triangles in the graph $\mathbf{y}$.



Other conditional independence statistics

⇒ Pattison and Robins (2002), Butts (2005)
⇒ Snijders, Pattison, Robins and Handcock (2004)

– edges in Y that are not tied are conditionally independent given the rest of the network

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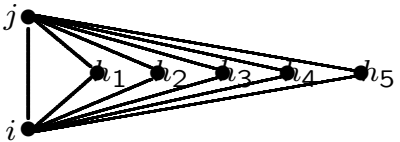
- k-triangle distribution: $\mathrm{t}_k(\mathbf{y}) =$ proportion of k-triangles in the graph $\mathbf{y}$.
- edgewise shared partner distribution:
  $\mathrm{p}_k(\mathbf{y}) =$ proportion of nodes with exactly k edgewise shared partners in $\mathbf{y}$.



k-triangle for $k = 5$, i.e., 5-triangle

Structural Signatures

– identify social constructs or features
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- Clusters of edges are often *transitive*:
  Include $t_1(y)$, the proportion of triangles amongst triads
  A closely related quantity is the *percent of complete triangles*
  or *mean clustering coefficient*

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Example: A simple model-class with transitivity

$n = 50$ actors $\quad N = 1225$ pairs $\quad 10^{369}$ graphs

$$P(Y = y) = \frac{\exp\{\eta_1 E(y) + \eta_2 C(y)\}}{\kappa(\eta_1, \eta_2)} \qquad y \in \mathcal{Y}$$

where

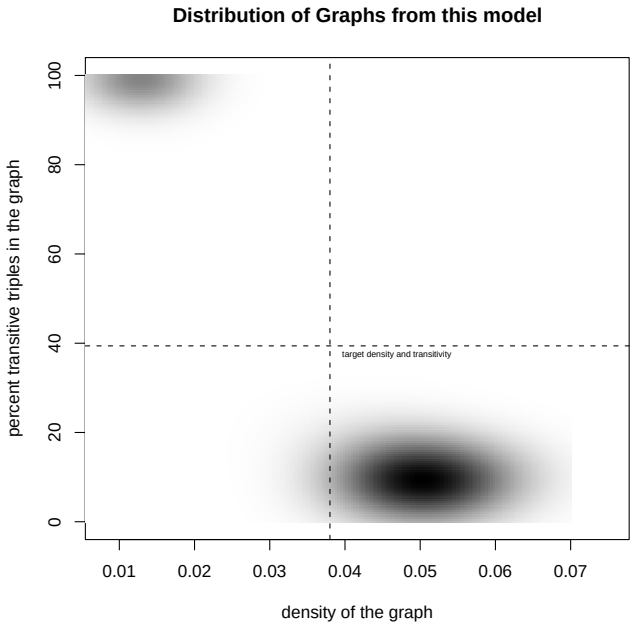
$E(x)$ is the density of edges $(0 - 1)$
$C(x)$ is the triangle percent $(0 - 100)$

- If we set the density of the graph to have about 50 edges then the expected triangle percent is 3.8%
- Suppose we set the triangle percent large to reflect transitivity in the graph: 38%

How can we tell if the model is useful?

- Does this model capture transitivity and density in a flexible way?

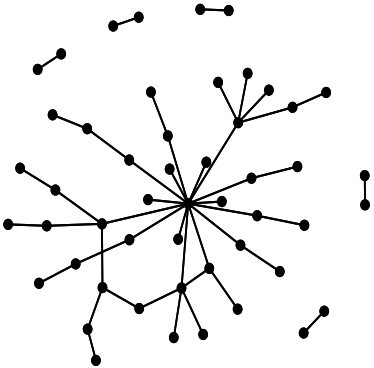
- By construction, on average, graphs from this model have average density 4% and average triangle percent 38%
- If the model is a good representation of transitivity and density we expect the graphs drawn from the model to be close to these values.
- What do graphs produced by this model look like?


Distribution of Graphs from this model
percent transitive triples in the graph
0
20
40
60
80
100
target density and transitivity
0.01
0.02
0.03
0.04
0.05
0.06
0.07
density of the graph

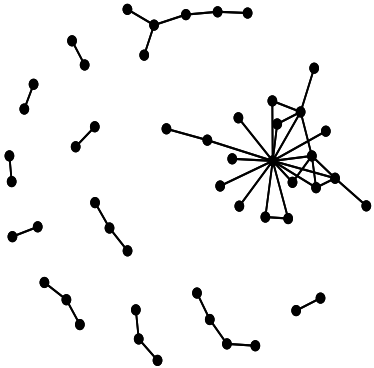
Illustrations of good models within this model-class

- village-level structure
  - $n = 50$
  - mean clustering coefficient = 15%
- larger-level structure
  - $n = 1000$
  - mean clustering coefficient = 15%
- Attribute mixing
  - Two-sex populations
  - mean clustering coefficient = 15%

Yule with zero clustering coefficient conditional on degree



Yule with clustering coefficient 15%



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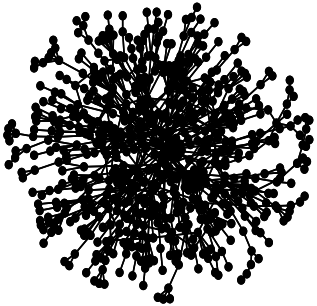
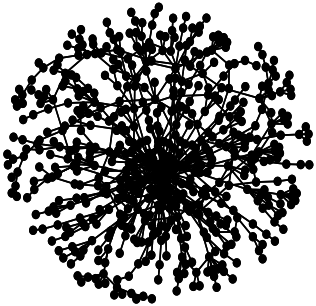
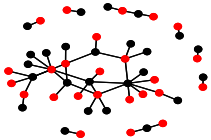


Yule with clustering coefficient 15%

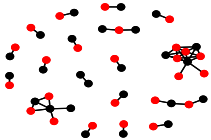


Heterosexual Yule with no correlation



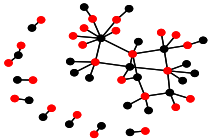
tripercent = 3

Heterosexual Yule with strong correlation



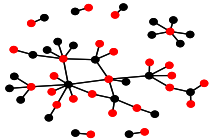
tripercent = 60.6

Heterosexual Yule with modest correlation



tripercent = 0

Heterosexual Yule with negative correlation



tripercent = 0

Conclusions and Challenges

- Models are a very constructive way to represent theory
- Homogeneity is a foundation to build models on
- Some seemingly simple models are not so.
- Useful models require additional development
- Simple models are being used to capture structural properties
- The inclusion of attributes is very important
  - actor attributes
  - dyad attributes e.g. homophily, race, location
  - structural terms e.g. transitive homophily

- latent class and trait models are important
  - an underlying latent “social space" of actors
    $\Rightarrow$ Hoff, Raftery and Handcock (2002)
    $\Rightarrow$ Hoff (2003, 2004 ,...)
  - latent class models are very promising
    $\Rightarrow$ Nowicki and Snijders (2001)
  - latent class and trait models
    $\Rightarrow$ Tantrum, Handcock, Raftery (2004)